Arbitrage Pricing Theory (APT)

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Basic assumptions

- The CAPM assumes homogeneous expectations and mean-variance preferences.
  - The result: The model identifies the market portfolio as the only risk factor
- The APT makes no assumption about expectations or investor risk preferences.
  - Consequently, the model does not identify any risk factor.
- The CAPM and the APT both require perfectly competitive securities markets

"No arbitrage opportunities"

- In the markets underlying both the CAPM and the APT, there are no opportunities for making arbitrage profits
  - This means that two securities with identical payoffs in all states must have the same price today ("Law of one price")
  - It also means that riskless investment opportunities earn the riskless rate of return.
  - Zero-investment, riskless cash flows are eliminated through arbitrage activity

"Linear pricing"

- It can be shown mathematically that the absence of arbitrage opportunities in the market implies that the expected return on any asset is a linear function of the expected return on priced risk factors
- Because the theory does not identify what the prices factors are, we posit an arbitrary set of K factors.
- The realized value of the k'th factor is F_k

$$(1) \quad r_i = E(r_i) + \beta_{i1} f_1 + \beta_{i2} f_2 + \ldots + \beta_{iK} f_K + e_i = E(r_i) + \sum_{k=1}^{K} \beta_{ik} f_k + e_i$$

- Equation (1) is a K-factor return generating process
- $E(r_i)$ = expected return on investment *i*
- f_k = k'th factor shock: F_k-$E(F_k)$, where F_k is the realized factor value and $E(f_k)=0$
- β_{ik} = security *i*'s sensitivity with respect to the k'th factor (factor loading, or factor risk)
- e_i = firm-specific risk

$$(2) \quad E(r_i) - r_F = \sum_{k=1}^{K} \beta_{ik} \lambda_k$$

- Equation (2) is the APT model
- r_F= the return on the risk-free asset
- λ_k= the risk premium of factor k (expected return on the k'th factor in excess of the risk-free return)

$$(3) \quad r_i - r_F = \sum_{k=1}^{K} \beta_{ik}(\lambda_k + f_k) + e_i$$

- Equation (3) combines (1) and (2)
- It says that the realized excess return on investment *i* is generated by *i*'s exposure to factor risks and the unexpected factor realization, plus firm-specific risk

- Problem:
  - Even if you know what the relevant factors F_k are, you probably do not know $E(F_k)$ and it is therefore also difficult to estimate f_k
- Solution: "factor mimicking"
  - Form a broad portfolio that has a β_k=1 on factor k while being independent of (β_k=0) the other K-1 factors
  - This is called a "factor mimicking portfolio"
  - Since this portfolio must also obey APT in (2), its expected excess return, $E(r_k)$, equals λ_k
  - Plug this back into equation (3) and you have a return generating process stated purely in terms of observables

- Mimicking portfolio for factor k
  - Portfolio k mimics a factor if
    - $\beta_k=1$, $\beta_{j\neq k}=0$
  - Price portfolio k using APT in (2):
    - $E(r_k)=r_F+\lambda_k$
  - Shocks to factor portfolio k:
    - $f_k = r_k - E(r_k) = r_k - (\lambda_k + r_F)$
  - Plug this shock into the right-hand side of the regression equation (3):
    - $\beta_{ik}(\lambda_k+f_k)=\beta_{ik}(\lambda_k+r_k-\lambda_k-r_F) = \beta_{ik}(r_k - r_F)$
  - Do the same for all K factors, and you get..

$$(3') \qquad r_i - r_F = \sum_{k=1}^{K} \beta_{ik}(r_k - r_F) + e_i$$

- Equation (3') says that the realized excess return on asset i is generated by i's sensitivity to the <u>realized</u> excess returns (i.e., the realized risk premiums) on a set of K factor-mimicking portfolios plus the firm-specific return realization

$$(4) \quad E(r_i) - r_F = \beta_{iM} \lambda_M = \beta_{iM} \left[E(r_M) - r_F \right]$$

- Equation (4) shows that the CAPM is a special case of the APT model (2) with only one factor, the return on value-weighted market portfolio M

$$(5) \quad (1 - \beta_{iM}) r_F + \beta_{iM} E(r_M) \equiv E(r_{Track})$$

- Equation (5) is obtained by rewriting the RHS of (4)
- This is a portfolio with weights (1-β_{iM}) in the risk-free asset and β_{iM} in the market M
- We call this portfolio a <u>tracking portfolio for investment i</u> since $E(r_i)=E(r_{Track})$
- This tracking portfolio is exposed to factor risk only

$E(r_i)=E(r_{Track})$

- What is the intuition behind this equation?
  - Suppose the equality does not hold. Then, sell short one and invest the proceeds in the other, generating a positive risk-free cash inflow with no net investment, i.e., an arbitrage profit.
  - Competition to capture this arbitrage profit restores the equality above
- Note: Investment i has a positive net present value only if $E(r_i) > E(r_{Track})$

$$(6) \quad E(r_{Track}) \equiv (1-\sum_{k=1}^{K}\beta_{ik})r_F + \sum_{k=1}^{K}\beta_{ik}(\lambda_k + r_F) = r_F + \sum_{k=1}^{K}\beta_{ik}\lambda_k = E(r_i)$$

- (6) generalizes (5) to K factors, where $\lambda_k + r_F$ replaces $E(r_k)$
  - The idea of a tracking portfolio is central to understanding how the market values an investment. In principle, any investment can be tracked
  - Thus, project valuation comes down to factor sensitivities of cash flows and of factor prices in an arbitrage-free environment

$$r_i = E(r_i) + \beta_{i1} f_1 + \beta_{i2} f_2 + e_i$$
$$r_A = .03 + 1 f_1 - 4 f_2 + e_A$$
$$r_B = .05 + 3 f_1 + 2 f_2 + e_B$$
$$r_C = .10 + 1.5 f_1 + 0 f_2 + e_C$$

- Example 1: Track the Wilshire 5000 stock index with stocks (A,B,C)
  - Suppose Wilshire has β_{W1}=2, β_{W2}=1

$$Factor\ 1: 1x_A + 3x_B + 1.5x_C = 2$$
$$Factor\ 2: -4x_A + 2x_B + 0x_C = 1$$
$$Portfolio\ : x_A + x_B + x_C = 1$$

- Find a set of portfolio weight for the tracking portfolio that satisfies the above system of three equations
  - The solution is x_A = -.1, x_B = .3, x_C = .8

Tracking with Pure Factor Portf.

- While the portfolio in example 1 tracks the Wilshire 5000, the tracking portfolio is likely to carry substantial firm-specific risk which the Wilshire 5000 does not
- The solution is to use well diversified portfolios as the underlying assets in the tracking portfolio

Pure factor Portfolios (PFPs)

- A PFP is a well diversified portfolio that tracks a given factor and that is also independent of all other factors
  - Thus, a PFP for factor *k* has β_{PFPk}=1 and β_{PFP}=0 for all the remaining K-1 factors
  - With K factors, one can use any set of K+1 investments that lack firm-specific risk (are well diversified) to create K PFPs

$$r_i = E(r_i) + \beta_{i1} f_1 + \beta_{i2} f_2 + e_i$$
$$r_c = .08 + 2 f_1 + 3 f_2 + e_c$$
$$r_g = .10 + 3 f_1 + 2 f_2 + e_g$$
$$r_s = .10 + 3 f_1 + 5 f_2 + e_s$$

- Example 2: Find the PFPs for f_1 and f_2 using the three stocks c,g,s
  - Need to solve two systems of linear equations, one for each FPF

$$PFP\ 1:$$
$$2x_c + 3x_g + 3x_s = 1$$
$$3x_c + 2x_g + 5x_s = 0$$
$$x_c + x_g + x_s = 1$$
$$\Rightarrow \quad x_c = 2, \quad x_g = 1/3, \quad x_s = -4/3$$
$$PFP\ 2:$$
$$2x_c + 3x_g + 3x_s = 0$$
$$3x_c + 2x_g + 5x_s = 1$$
$$x_c + x_g + x_s = 1$$
$$\Rightarrow \quad x_c = 3, \quad x_g = -2/3, \quad x_s = -4/3$$

- What are the factor equations for the PFPs?

$$r_{PFPk} = E(r_{PFPk}) + f_k \qquad k = 1,2$$

$$E(r_{PFPi}) = \sum_{n=1}^{N} x_{PFPi,n} E(r_n)$$

$$E(r_{PFP1}) = (2)(.08) + (1/3)(.10) - (4/3)(.10) = .06$$

$$E(r_{PFP2}) = (3)(.08) - (2/3)(.10) - (4/3)(.10) = .04$$

Thus:

$$r_{PFP1} = .06 + f_1$$

$$r_{PFP2} = .04 + f_2$$

Also, compute factor premiums from equation (6)

$$E(r_{PFPk}) \equiv (1 - \sum_{k=1}^{K} \beta_{PFPk}) r_F + \sum_{k=1}^{K} \beta_{PFPk} (\lambda_k + r_F) = \lambda_k + r_F$$

$$\text{If } r_F = .05, \quad \text{then} \quad \lambda_1 = .01, \quad \lambda_2 = -.01$$

- So, you track an investment with β_1=.25, β_2=.5 simply by holding a portfolio consisting of PFP1, PFP2 and the risk-free asset, with portfolio weights
  - x_{PFP1}=.25
  - x_{PFP2}=.5
  - x_F=1-(.25+.5)=.25
- In general, with K factors, an investment with no firm-specific risk and a factor beta of β_k on the k'th factor is tracked by a portfolio with weights
  - β_k on the pure factor portfolio for *k* and
  - (1- $\sum_k \beta_k$) on the risk-free asset

- With no firm-specific risk, the factor equations of the tracking portfolio and the tracked investment will be identical with the possible exception of the intercept terms
- Differences in the intercept terms represent differences in expected returns
- If the expected returns differ, short the one with the smallest intercept, and go long in the other
- This generates arbitrage profits (positive net present value) for the investment

Factor Models and Asset Allocation

- Suppose there are 8,000 traded stocks
- The full variance-covariance matrix of these stocks contains 64 million elements
- The use of factor models greatly simplifies the estimation of these variances and covariances

The simplification occurs because the factors are uncorrelated with each other and with firm-specific risks (the variance-covariance matrix is diagonal). Moreover, firm-specific risks are uncorrelated across individual securities. We have that:

(1) $$r_i = E(r_i) + \beta_{i1} f_1 + \beta_{i2} f_2 + ... + \beta_{iK} f_K + e_i = E(r_i) + \sum_{k=1}^{K} \beta_{ik} f_k + e_i$$

(7) $$Var(r_i) = \sum_{k=1}^{K} \beta_{ik}^2 Var(F_k) + Var(e_i)$$

(8) $$Cov(r_i, r_j) = \sum_{k=1}^{K} \beta_{ik} \beta_{jk} Var(F_k)$$

- Thus, with 8,000 stocks and, e.g., five factors, you need only to estimate 40,000 stock betas, five factor variances and 8,000 firm-specific variances to reproduce the full variance-covariance matrix
- This explains the popularity of factor models in asset allocation and investment decisions in practice
- For well diversified portfolios, the firm-specific variance component is close to zero, so that the total variance is driven purely by factor risk

Identifying the Factors

- The APT does not identify the factors
- One method to identify factors is to apply principal-components analysis to the variance-covariance matrix of security returns. The resulting factors are, however, difficult to interpret and may have "strange" portfolio weights
- Our approach: Use economic theory to prespecify a set of "reasonable" macro-economic factors and see what works

Frequently used macro-economic factors:

- The market index
- Unexpected growth in industrial production (or unexpected changes in the business cycle)
- Changes in expected and unexpected inflation (changes in expected inflation proxied by the change in the T-bill rate)
- Unexpected changes in default spread: unexpected changes in the spread between AAA-rated and BAA-rated corporate bond returns
- Unexpected changes in the term spread as measured by the difference between long and short government bonds